On the Definition of Reducible Hypercomplex Number Systems.

II.

By Heman Burr Leonard.

§1.—*Preliminary.*—The present paper is intended as a completion of the problem studied in a former paper bearing the same title.*

A hypercomplex number system is said to be reducible when, by a proper choice of units, it can be brought to the form

$$E \equiv E_j\, E_k \equiv e_1 \,\ldots\ldots\, e_m e_{m+1} \,\ldots\ldots\, e_n,$$

where the following conditions are fulfilled:

A), associativity of E;
C_1), E_j forms a system by itself;
C_2), E_k forms a system by itself;
C_{jk}), $e_j e_k = 0, \qquad j = 1, \ldots\ldots, m;$
C_{kj}), $e_k e_j = 0, \qquad k = m+1, \ldots\ldots, n.$

In the former paper were listed seventy-eight different definitions from which the above requirements can be deduced, and these definitions were based on the following twenty conditions:

A_1), $\qquad (J_1\, J_2)\, J_3 = J_1\, (J_2\, J_3);$
A_2), $\qquad (K_1 K_2) K_3 = K_1 (K_2 K_3);$
A_3), $\qquad (K_1\, J_1)\, J_2 = K_1\, (J_1\, J_2);$
A_4), $\qquad (J_1\, K_1)\, K_2 = J_1\, (K_1\, K_2);$
A_5), $\qquad (J_1\, K_1)\, J_2 = J_1\, (K_1\, J_2);$
A_6), $\qquad (K_1 J_1)\, K_2 = K_1\, (J_1\, K_2);$
A_7), $\qquad (J_1\, J_2)\, K_1 = J_1\, (J_2\, K_1);$
A_8), $\qquad (K_1\, K_2)\, J_1 = K_1\, (K_2\, J_1);$
C_1), E_j is closed under multiplication, that is, $J_1\, J_2 = J_3$;
C_2), E_k is closed under multiplication, that is, $K_1\, K_2 = K_3$;

* Epsteen-Leonard, On the Definition of Reducible Hypercomplex Number Systems, American Mathematics, Vol. XXVII (1905), p. 217.

C^j_{jk}), $\quad J_1 K_1 = K_2 \quad (J_2 = 0)$;

C^k_{jk}), $\quad J_1 K_1 = J_2 \quad (K_2 = 0)$;

C^j_{kj}), $\quad K_1 J_1 = K_2 \quad (J_2 = 0)$;

C^k_{kj}), $\quad K_1 J_1 = J_2 \quad (K_2 = 0)$;

C_r), right-hand division possible and unique, that is, not every X is a right-hand divisor of zero; hence, an X exists such that

$$X_1 X = 0, \text{ only if } X_1 = 0;$$

C_l), left-hand division possible and unique, that is, not every X is a left-hand divisor of zero; hence, an X exists such that

$$X X_1 = 0, \text{ only if } X_1 = 0;$$

C^j_r), right-hand division possible and unique in the subset E_j, that is, not every J is a right-hand divisor of zero; hence, a J exists such that

$$J_1 J = 0, \text{ only if } J_1 = 0;$$

C^j_l), left-hand division possible and unique in the subset E_j, that is, not every J is a left-hand divisor of zero; hence, a J exists such that

$$J J_1 = 0, \text{ only if } J_1 = 0;$$

C^k_r), right-hand division possible and unique in the subset E_k, that is, not every K is a right-hand divisor of zero; hence, a K exists such that

$$K_1 K = 0, \text{ only if } K_1 = 0;$$

C^k_l), left-hand division possible and unique in the subset E_k, that is, not every K is a left-hand divisor of zero; hence, a K exists such that

$$K K_1 = 0, \text{ only if } K_1 = 0.$$

Conditions A_1, A_2, A_3, A_4, A_5, A_6, A_7, and A_8, together are equivalent to the associativity condition

$$A), \quad (X_1 X_2) X_3 = X_1 (X_2 X_3),$$

where X_p $(p = 1, 2, 3)$ are any numbers of the system E and where

$$X_p = J_p + K_p = \sum_{j_1=1}^{m} x_{pj_1} e_{j_1} + \sum_{k_1=m+1}^{n} x_{pk_1} e_{k_1}.$$

It was shown also that the conditions composing each definition are independent.

By systematic checking I have found that the ninety-six definitions given in the following tables, together with the seventy-eight of the former paper, give all possible methods of defining reducibility that depend upon the above twenty assumptions.

In this paper I use throughout the notation and the tables of the former paper; the definitions being based of course upon *Table I.* *Table II.* being complete, no additions are made; *Tables* III_2. and IV_2. of this paper supplement *Tables III.* and *IV.* of the former paper. In the notation R_{13_1}, the subscript indicates that this is the first definition which is to be inserted after R_{13} of the former paper to form the sequence as now completed. Definition R_{13_2} immediately follows R_{13_1}.

Independence Proofs.—With the exception of those indicated by a (*) in the tables, the conditions composing the various definitions are independent. The independence proofs are easily verified by *Table V.*, which is reprinted from the former paper.

§2.—*Definitions of Reducibility by Independent Assumptions.*

We reproduce *Table I.* from the former paper.

TABLE I.

Notation.	Assumptions.											Consequence.				
D_1	A_7,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},	C_r										C_1
D_2	A_3,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},		C_l									C_1
D_3	A_7,	C^j_{jk},	C^k_{jk},							C^k_r						C_1
D_4	A_3,			C^j_{kj},	C^k_{kj},						C^k_l					C_1
D_5	A_8,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},	C_r										C_2
D_6	A_4,	C^j_{jk},	C^k_{jk},	C^j_{kj},	C^k_{kj},		C_l									C_2
D_7	A_8,			C^j_{kj},	C^k_{kj}			C^j_r								C_2
D_8	A_4,	C^j_{jk},	C^k_{jk},						C^j_l							C_2
D_9	A_5,			C^j_{kj},	C^k_{kj},			C^j_r				C^j_{jk}				
D_{10}	A_6,			C^j_{kj},	C^k_{kj},						C^k_l		C^k_{jk}			
D_{11}	A_6,	C^j_{jk},	C^k_{jk},							C^k_r				C^k_{kj}		
D_{12}	A_5,	C^j_{jk},	C^k_{jk},						C^j_l						C^j_{kj}	

I.—From a consideration of this table, it is evident that D_6, D_5, D_2, D_1 are the only dependencies in which the division assumptions are on the system

E as a whole. There are eight possible combinations of these that give a definition of reducibility and they all appear in *Table II.*

II.—In D_{12}, D_{11}, D_{10}, D_9, D_8, D_7, D_4, and D_3 the division assumptions are on the subsets E_j and E_k. The thirty-eight definitions of *Table III.* with the additional twenty-four of *Table III$_2$.* give their sixty-two possible combinations.

TABLE III$_2$.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
R_{13_1}	D_7	A_8 C^j_r	" "		C_2
	....	C_1			
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{13_2}	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_4	A_3 C^k_l	" "		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{15_1}	....	C^k_{kj}			
	D_8	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_4	A_3 C^k_{kj} C^k_l	"		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{17_1}	....	C^k_{kj}			
	D_7	A_8 C^k_{kj} C^j_r	"		C_2
	D_4	A_3 C^k_{kj} C^k_l	"		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{17_2}	....	C^k_{kj}			
	D_7	A_8 C^k_{kj} C^j_r	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	

TABLE III_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{17_3}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_7	A_8 C^k_{kj} C^j_r	"		C_2
	....	C_1			
R_{17_4}	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_2			
	D_4	A_3 C^k_{kj} C^k_l	"		C_1
R_{19_1}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_8	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_4	A_3 C^j_{kj} C^k_l	"		C_1
R_{21_1}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} C^j_r	"		C_2
	D_4	A_3 C^j_{kj} C^k_l	"		C_1
R_{21_2}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} C^j_r	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	
R_{21_3}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} C^j_r	"		C_2
	....	C_1			
R_{21_4}	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_4	A_3 C^j_{kj} C^k_l	"		C_1

TABLE III_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	(5)	(6) Proved by (3) and (4).
R_{25_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_8	A_4 C^j_l	" "		C_2
	. . . .	C_1			
R_{29_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	. . . .	C_2			
	D_3	A_7 C^k_r	" "		C_1
R_{30_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	. . . .	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	D_4	A_3 C^j_{kj} C^k_{kj} C^k_l		C_1	
R_{30_2}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	. . . .	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	D_3	A_7 C^j_{jk} C^k_r	"		C_1
R_{30_3}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	. . . .	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	. . . .	C_1			
R_{31_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	. . . .	C^j_{jk}			
	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_2	
	D_3	A_7 C^j_{jk} C^k_r	"		C_1

TABLE III_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.											Proved by (3): (4)	Proved by (3): (5)	(6) Proved by (3) and (4).
	D_{10}	A_6					C^j_{kj}	C^k_{kj}				C^k_l	C^k_{jk}		
R_{33_1}	....				C^j_{jk}										
	....			C_2											
	D_3	A_7			C^j_{jk}						C^k_r		"		C_1
	....					C^k_{jk}									
R_{34_1}	D_9	A_5					C^j_{kj}	C^k_{kj}	C^j_r				C^j_{jk}		
	D_8	A_4				C^k_{jk}				C^j_l			"		C_2
	D_4	A_3					C^j_{kj}	C^k_{kj}				C^k_l		C_1	
	....					C^k_{jk}									
R_{34_2}	D_9	A_5					C^j_{kj}	C^k_{kj}	C^j_r				C^j_{jk}		
	D_8	A_4				C^k_{jk}				C^j_l			"		C_2
	D_3	A_7				C^k_{jk}					C^k_r		"		C_1
	....					C^k_{jk}									
R_{34_3}	D_9	A_5					C^j_{kj}	C^k_{kj}	C^j_r				C^j_{jk}		
	D_8	A_4				C^k_{jk}				C^j_l			"		C_2
	....		C_1												
	....					C^k_{jk}									
R_{35_1}	D_9	A_5					C^j_{kj}	C^k_{kj}	C^j_r				C^j_{jk}		
	D_7	A_8					C^j_{kj}	C^k_{kj}	C^j_r					C_2	
	D_3	A_7				C^k_{jk}					C^k_r		"		C_1
	....					C^k_{jk}									
R_{37_1}	D_9	A_5					C^j_{kj}	C^k_{kj}	C^j_r				C^j_{jk}		
	....			C_2											
	D_3	A_7				C^k_{jk}					C^k_r		"		C_1

III.—The thirty-two definitions of *Table IV.* and the additional seventy-two of *Table* IV_2. each contain at least one division assumption upon the system E as a whole and at least one division assumption on one of its subsets.

TABLE IV_2.

(1) Nota-tion.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{54_1}	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_6	A_4 C^j_{jk} C^k_{jk} C_l	" "		C_2
	....	C_1			
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{58_1}	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C_r	" "		C_2
	....	C_1			
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
R_{58_2}	....	C_2			
	D_2	A_3 C^j_{jk} C^k_{jk} C_l	" "		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{58_3}	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_1	A_7 C^j_{jk} C^k_{jk} C_r	" "		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
R_{58_4}	D_8	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{58_5}	....	C^k_{kj}			
	D_8	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_1	A_7 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{58_6}	....	C^k_{kj}			
	D_7	A_8 C^k_{kj} C^j_r	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{58_7}	....	C^k_{kj}			
	D_7	A_8 C^k_{kj} $*C^j_r$	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{58_8}	....	C^k_{kj}			
	D_6	A_4 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_2
	D_4	A_3 C^k_{kj} $*C^k_l$	"		C_1
	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
R_{58_9}	....	C^k_{kj}			
	D_6	A_4 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{58_{10}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_6	A_4 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1
$R_{58_{11}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_6	A_4 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_1
$R_{58_{12}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_6	A_4 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_2
	....	C_1			
$R_{58_{13}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	D_4	A_3 C^k_{kj} C^k_l	"		C_1
$R_{58_{14}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} $*C^k_r$		C_1	

TABLE IV_2.—Continued.

(1) Nota-tion.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{58_{15}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1
$R_{58_{16}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_1
$R_{58_{17}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	D_5	A_8 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_2
	....	C_1			
$R_{58_{18}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_2			
	D_2	A_3 C^j_{jk} C^k_{jk} C^k_{kj} C_l	"		C_1
$R_{58_{19}}$	D_{12}	A_5 C^j_{jk} C^k_{jk} C^j_l	C^j_{kj}		
	....	C^k_{kj}			
	....	C_2			
	D_1	A_7 C^j_{jk} C^k_{jk} C^k_{kj} C_r	"		C_1

TABLE IV$_2$.—Continued.

(1) Nota-tion.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	....	C^j_{kj}			
$R_{58_{20}}$	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_8	A_4 C^j_{jk} C^k_{jk} $*C^j_l$		C_2	
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_1
	....	C^j_{kj}			
$R_{58_{21}}$	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_8	A_4 C^j_{jk} C^k_{jk} C^j_l		C_2	
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
	....	C^j_{kj}			
$R_{58_{22}}$	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} C^j_r	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_1
	....	C^j_{kj}			
$R_{58_{23}}$	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_7	A_8 C^j_{kj} $*C^j_r$	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
	....	C^j_{kj}			
$R_{58_{24}}$	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_4	A_3 C^j_{kj} $*C^k_l$	"		C_1

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{25}}$	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{26}}$	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_2	A_3 C^j_{jk} C^k_{ik} C^j_{kj} C_l	"		C_1
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{27}}$	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{28}}$	D_6	A_4 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_2
	....	C_1			
	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
$R_{58_{29}}$	D_5	A_8 C^j_{ik} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_4	A_3 C^j_{kj} C^k_l	"		C_1

TABLE IV_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{58_{30}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_3	A_7 C^j_{jk} C^k_{jk} C^k_r		C_1	
$R_{58_{31}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_1
$R_{58_{32}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
$R_{58_{33}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	D_5	A_8 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_2
	....	C_1			
$R_{58_{34}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_2	A_3 C^j_{jk} C^k_{jk} C^j_{kj} C_l	"		C_1

TABLE IV$_2$.—Continued.

(1) Nota-tion.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{58_{35}}$	....	C^j_{kj}			
	D_{11}	A_6 C^j_{jk} C^k_{jk} C^k_r	C^k_{kj}		
	....	C_2			
	D_1	A_7 C^j_{jk} C^k_{jk} C^j_{kj} C_r	"		C_1
R_{66_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_6	A_4 C^j_{kj} C^k_{kj} C_l	" "		C_2
	....	C_1			
R_{70_1}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_5	A_8 C^j_{kj} C^k_{kj} C_r	" "		C_2
	....	C_1			
R_{70_2}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	....	C_2			
	D_2	A_3 C^j_{kj} C^k_{kj} C_l	" "		C_1
R_{70_3}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	....	C_2			
	D_1	A_7 C^j_{kj} C^k_{kj} C_r	" "		C_1

TABLE IV_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
R_{70_4}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_8	A_4 C^j_{jk} $*C^j_l$	"		C_2
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
R_{70_5}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_8	A_4 C^j_{jk} C^j_l	"		C_2
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
R_{70_6}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_2	
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
R_{70_7}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_7	A_8 C^j_{kj} C^k_{kj} $*C^j_r$		C_2	
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
R_{70_8}	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_4	A_3 C^j_{kj} C^k_{kj} C^k_l		C_1	

TABLE IV_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
R_{70_9}	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_3	A_7 C^j_{jk} C^k_r	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{10}}$	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{11}}$	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{12}}$	D_6	A_4 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_2
	....	C_1			
	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
$R_{70_{13}}$	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_4	A_3 C^j_{kj} C^k_{kj} C^k_l		C_1	

TABLE IV_2—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{70_{14}}$	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_3	A_7 C^j_{jk} $*C^k_r$	"		C_1
$R_{70_{15}}$	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
$R_{70_{16}}$	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
$R_{70_{17}}$	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	D_5	A_8 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	....	C_1			
$R_{70_{18}}$	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	....	C_2			
	D_2	A_3 C^j_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
$R_{70_{19}}$	D_{10}	A_6 C^j_{kj} C^k_{kj} C^k_l	C^k_{jk}		
	....	C^j_{jk}			
	....	C_2			
	D_1	A_7 C^j_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
$R_{70_{20}}$	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_8	A_4 C^k_{jk} $*C^j_l$	"		C_2
	D_2	A_3 C^k_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
$R_{70_{21}}$	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_8	A_4 C^k_{jk} C^j_l	"		C_2
	D_1	A_7 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
$R_{70_{22}}$	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_2	
	D_2	A_3 C^k_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
$R_{70_{23}}$	....	C^k_{jk}			
	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_7	A_8 C^j_{kj} C^k_{kj} C^j_r		C_2	
	D_1	A_7 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1

TABLE IV$_2$.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.		Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	....		C_{jk}^{k}			
	D_9	A_5	$C_{kj}^{j}\ C_{kj}^{k}\quad C_r^j$	C_{jk}^{j}		
$R_{70_{24}}$	D_6	A_4	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\quad C_l$	"		C_2
	D_4	A_3	$C_{kj}^{j}\ C_{kj}^{k}\quad *C_l^k$		C_1	
	....		C_{jk}^{k}			
	D_9	A_5	$C_{kj}^{j}\ C_{kj}^{k}\quad C_r^j$	C_{jk}^{j}		
$R_{70_{25}}$	D_6	A_4	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\quad C_l$	"		C_2
	D_3	A_7	$C_{jk}^{k}\quad C_r^k$	"		C_1
	....		C_{jk}^{k}			
	D_9	A_5	$C_{kj}^{j}\ C_{kj}^{k}\quad C_r^j$	C_{jk}^{j}		
$R_{70_{26}}$	D_6	A_4	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\quad C_l$	"		C_2
	D_2	A_3	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\quad C_l$	"		C_1
	....		C_{jk}^{k}			
	D_9	A_5	$C_{kj}^{j}\ C_{kj}^{k}\quad C_r^j$	C_{jk}^{j}		
$R_{70_{27}}$	D_6	A_4	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\quad C_l$	"		C_2
	D_1	A_7	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\ C_r$	"		C_1
	....		C_{jk}^{k}			
	D_9	A_5	$C_{kj}^{j}\ C_{kj}^{k}\quad C_r^j$	C_{jk}^{j}		
$R_{70_{28}}$	D_6	A_4	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\quad C_l$	"		C_2
	....	C_1				
	....		C_{jk}^{k}			
	D_9	A_5	$C_{kj}^{j}\ C_{kj}^{k}\quad C_r^j$	C_{jk}^{j}		
$R_{70_{29}}$	D_5	A_8	$C_{jk}^{k}\ C_{kj}^{j}\ C_{kj}^{k}\ C_r$	"		C_2
	D_4	A_3	$C_{kj}^{j}\ C_{kj}^{k}\quad C_l^k$		C_1	

TABLE IV_2.—Continued.

(1) Notation.	(2) From Table I.	(3) Assumptions.	Proved by (3) (4)	Proved by (3) (5)	(6) Proved by (3) and (4).
	....	C^k_{jk}			
$R_{70_{30}}$	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_5	A_8 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_3	A_7 C^k_{jk} $*C^k_r$	"		C_1
	....	C^k_{jk}			
$R_{70_{31}}$	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_5	A_8 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_2	A_3 C^k_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
	....	C^k_{jk}			
$R_{70_{32}}$	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_5	A_8 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	D_1	A_7 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1
	....	C^k_{jk}			
$R_{70_{33}}$	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	D_5	A_8 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_2
	....	C_1			
	....	C^k_{jk}			
$R_{70_{34}}$	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	....	C_2			
	D_2	A_3 C^k_{jk} C^j_{kj} C^k_{kj} C_l	"		C_1
	....	C^k_{jk}			
$R_{70_{35}}$	D_9	A_5 C^j_{kj} C^k_{kj} C^j_r	C^j_{jk}		
	....	C_2			
	D_1	A_7 C^k_{jk} C^j_{kj} C^k_{kj} C_r	"		C_1

§3.—*Independence Proofs.*

The sets of conditions in *Tables* III_2. and IV_2. yield reducibility. It can readily be seen that there are not enough assumptions in any one of these sets to prove the associativity A of the system E. In each case this can be accomplished by adding the assumptions A_1 and A_2. By the use of *Table* V. the mutual independence of the conditions composing each of the definitions of *Table* III_2. and those of all but twelve of the definitions of *Table* IV_2. can be established. The independencies of *Table* V. are based upon the following multiplication tables, for which the borders have been omitted.

I.

$$\begin{array}{ccc} e_2 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2; E_k = e_3.$

II.

$$\begin{array}{cc} e_1 + e_2 & 0 \\ 0 & e_2 \end{array}$$

$E_j = e_1; E_k = e_2.$

III.

$$\begin{array}{ccc} e_1 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E = e_1, e_2; E_k = e_3.$

IV.

$$\begin{array}{ccc} e_1 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1; E_k = e_2, e_3.$

V.

$$\begin{array}{cc} e_1 & e_1 \\ e_1 & e_2 \end{array}$$

$E = e_1; E_k = e_2.$

VI.

$$\begin{array}{ccc} 0 & 0 & e_1 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2; E_k = e_3.$

VII.

$$\begin{array}{ccc} e_1 & 0 & e_3 \\ 0 & e_2 & 0 \\ 0 & e_3 & 0 \end{array}$$

$E_j = e_1; E_k = e_2, e_3.$

VIII.

$$\begin{array}{ccc} e_1 & 0 & 0 \\ e_2 & 0 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2; E_k = e_3.$

IX.

$$\begin{array}{ccc} 0 & 0 & 0 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{array}$$

$E_j = e_1, e_2; E_k = e_3.$

TABLE V.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	C_1	C_2	C^j_{jk}	C^k_{jk}	C^j_{kj}	C^k_{kj}	C_l	C_r	C^j_l	C^k_l	C^j_r	C^k_r	Proof.
1	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	I.
2	★	i	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (1).
3	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	★	II.
4	★	★	★	i_1	★	★	★	i_2	★	i_3	★	★	★	★	★	★	★	★	★	★	Interchange j, k in (3).
5	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	★	III.
6	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	★	IV.
7	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	★	★	★	★	★	★	Interchange j, k in (6).
8	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	★	★	★	★	★	Interchange j, k in (5).
9	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	★	V.
10	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	★	★	i_2	★	VI.
11	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	★	★	★	Interchange j, k in (9).
12	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	★	★	VII.
13	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	★	★	★	Interchange j, k in (12).
14	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	★	★	i_2	Interchange j, k in (10).
15	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	i_2	★	★	★	VIII.
16	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	★	Interchange j, k in (15).
17	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	i_2	★	IX.
18	★	★	★	★	★	★	★	★	★	★	★	★	★	★	★	i_1	★	★	★	i_2	Interchange j, k in (17).

§4.—*Dependence Proofs.*

The twelve sets of conditions of *Table* IV_2. that require individual consideration are R_{58_7}, R_{58_8}, $R_{58_{14}}$, $R_{58_{20}}$, $R_{58_{23}}$, $R_{58_{24}}$, R_{70_4}, R_{70_7}, $R_{70_{14}}$, $R_{70_{20}}$, $R_{70_{24}}$, and $R_{70_{30}}$.

1. That C^j_r is a consequence of the other conditions of R_{58_7} can be shown in the following manner. The conditions from which C^j_{kj} is derived in D_{12} are independent. By D_1 and D_5 it is seen that C_1 and C_2 are consequences of A_7, A_8, C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj}, C_r and all these conditions are mutually independent. According to the condition C_r, there exist a J and a K such that $(J_1 + K_1)(J + K) = 0$,

only if $J_1 = 0 = K_1$. Multiplying out, we have in view of C^j_{jk}, C^k_{jk}, C^j_{kj}, C^k_{kj} that

$$J_1 J + K_1 K = 0, \text{ only if } J_1 = 0 = K_1$$

or

$$J_3 + K_3 = 0, \qquad \text{by } C_1, C_2.$$

By the linear independence of the units, it follows that

$$\left.\begin{array}{l} J_3 = J_1 J = 0 \\ K_3 = K_1 K = 0 \end{array}\right\} \text{only if } \left\{\begin{array}{l} J_1 = 0 \\ K_1 = 0. \end{array}\right.$$

The former condition is C^j_r.*

2. That C^k_l is a consequence of the other conditions of R_{58_8} can be shown by the successive use of D_{12}, D_6, D_2, and C_l.

3. That C^k_r is a consequence of the other conditions of $R_{58_{14}}$ can be shown by the successive use of D_{12}, D_5, D_1, and C_r.

4. That C^j_l is a consequence of the other conditions of $R_{58_{20}}$ can be shown by the successive use of D_{11}, D_6, D_2, and C_l.

5. That C^j_r is a consequence of the other conditions of $R_{58_{23}}$ can be shown by the successive use of D_{11}, D_5, D_1, and C_r.

6. That C^k_l is a consequence of the other conditions of $R_{58_{24}}$ can be shown by the successive use of D_{11}, D_6, D_2, and C_l.

7. That C^j_l is a consequence of the other conditions of R_{70_4} can be shown by the successive use of D_{10}, D_6, D_2, and C_l.

8. That C^j_r is a consequence of the other conditions of R_{70_7} can be shown by the successive use of D_{10}, D_5, D_1, and C_r.

9. That C^k_r is a consequence of the other conditions of $R_{70_{14}}$ can be shown by the successive use of D_{10}, D_5, D_1, and C_r.

---

*In §5 of the previous paper the corresponding dependence theorems are correctly given. Some of the proofs, however, contain a weakness. Thus, in the above proof, it is not correct to continue the argument: Multiplying on the left by J', since

$$J_1 J + K_1 K = 0, \text{ only if } J_1 = 0 = K_1, \text{ and since } J_1 J = J_3 \text{ and } K_1 K = K_3, \text{ then}$$
$$J'(J_1 J + K_1 K) = 0, \text{ only if } J_1 = 0 = K_1 \text{ and therefore}$$
$$J'(J_1 J + K_1 K) = 0, \text{ only if } J_1 J = 0 = K_1 K,$$

which gives the required condition.

The reason that this argument is not valid is that the *product* J_3 ($= J_1 J$), although different from zero, may be a right-hand divisor of zero.

By the method used in the body of this paper all the dependence proofs in the previous paper can easily be modified so as to be correct.

10. That C_l^j is a consequence of the other conditions of $R_{70_{20}}$ can be shown by the successive use of D_9, D_6, D_2, and C_l.

11. That C_l^k is a consequence of the other conditions of $R_{70_{24}}$ can be shown by the successive use of D_9, D_6, D_2, and C_l.

12. That C_r^k is a consequence of the other conditions of $R_{70_{30}}$ can be shown by the successive use of D_9, D_5, D_1, and C_r.

UNIVERSITY OF COLORADO, *September*, 1905.